Supplemental Material

Micro responses to macro shocks

Martín Almuzara[*] Víctor Sancibrián[†]

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[Link to the paper]

B Additional proofs

We adopt the following notation in the proofs below. We use P_N, E_N, Var_N, Cov_N to denote probability, expectation, variance and covariance given $\{\theta_i, s_i\}_{i=1}^N$ (we insert a subindex κ or κ_T when necessary).

With a slight abuse of nomenclature we sometimes call Loève's inequality to the statement $|\sum_{i=1}^m X_i|^r \leq c_r \sum_{i=1}^m |X_i|^r$ (with $c_r = 1$ if $r \leq 1$ and $c_r = m^{r-1}$ otherwise) where $X_1, \ldots, X_m$ are random variables and not just to $E\left[|\sum_{i=1}^m X_i|^r\right] \leq c_r \sum_{i=1}^m E\left[|X_i|^r\right]$ (which is implied by the former). See Davidson (1994, Theorem 9.28).

Without loss of generality we assume $\kappa \geq 0$. We also define the scaling function $g(\kappa) = \max\{1, \kappa\}$ and note that $g(\kappa)/\kappa = g(\kappa^{-1})$. In Proposition 1

$$\frac{V(h,\kappa)}{g(\kappa^2/N)} = \frac{\sum_{\ell=0}^{\infty}\left\{\iota_\ell(h)\bar{\beta}_\ell^2 E_N\left[X_t^2 X_{t+h-\ell}^2\right] + \bar{\gamma}_\ell^2 E_N\left[X_t^2 Z_{t+h-\ell}^2\right]\right\}}{g(\kappa^2/N)} + \frac{\sum_{i=1}^N \sum_{\ell=0}^{\infty} \hat{s}_i^2 \delta_{i\ell}^2 E_N\left[X_t^2 u_{i,t+h-\ell}^2\right]}{N g(N/\kappa^2)}$$

is bounded below by $\underline{\text{CM}}^2 > 0$ and above by $3C^4 M_4 < \infty$ for any κ (and h). The same applies to $V(h,\kappa)/g(\kappa^2/N)$ in Proposition 2. In Proposition 3, $\text{tr}\{\boldsymbol{V}(h,\kappa)\}/g(\kappa^2/N)$ is bounded below by $(a_0^2+1)\underline{\text{CM}}^2 > 0$ and above by $6(p+1)(a_0^2+1)C^4 M_4 < \infty$.

[*]Federal Reserve Bank of New York: martin.almuzara@ny.frb.org

[†]CEMFI: victor.sancibrian@cemfi.edu.es

Proposition 1

Parts (A), (B) and (C) of the proof of Proposition 1 in Appendix A are established in Lemmas 1, 2 and 3 below. Lemmas 4 and 5 provide auxiliary results for Lemma 1, while 6 and 7 do the same for 2. At all times, we make Assumptions 1, 2 and 3 and we fix h and $p \geq h$ as $T, N \to \infty$ (note we do not need $T/N \to 0$ here).

Lemma 1 (Asymptotic normality of the score).

$$\frac{\sum_{t=1}^{T-h} X_t \xi_t(h, \kappa_T)}{\sqrt{(T-h)V(h, \kappa_T)}} \xrightarrow[P_{\kappa_T}]{\text{d}} N(0, 1).$$

Proof. The argument relies on the martingale representation:

$$\sum_{t=1}^{T-h} \frac{X_t \xi_t(h, \kappa_T)}{\sqrt{(T-h)V(h, \kappa_T)}} = \sum_{t=1}^{T} \chi_{T,t}(h, \kappa_T)$$

where we have defined

$$\chi_{T,t}(h, \kappa) = \frac{X_t \Xi_{X,t}(h, \kappa) + Z_t \Xi_{Z,t}(h) + (\kappa_T/N) \sum_{i=1}^{N} u_{it} \Xi_{U,it}(h)}{\sqrt{(T-h)V(h, \kappa_T)}}$$

with

$$\begin{aligned}
\Xi_{X,t}(h, \kappa) &= \sum_{\ell=1}^{h} \mathbb{1}\{t-\ell \geq 1\} \bar{\beta}_{h-\ell} X_{t-\ell} + \sum_{\ell=p+1}^{\infty} \mathbb{1}\{t \leq T-h\} \bar{\beta}_{h+\ell} X_{t-\ell} \\
&\quad + \sum_{\ell=0}^{\infty} \mathbb{1}\{t \leq T-h\} \left[\bar{\gamma}_{h+\ell} Z_{t-\ell} + \frac{\kappa}{N} \sum_{i=1}^{N} \hat{s}_i \delta_{i,h+\ell} u_{i,t-\ell} \right], \\
\Xi_{Z,t}(h) &= \sum_{\ell=1}^{h} \mathbb{1}\{t-\ell \geq 1\} \bar{\gamma}_{h-\ell} X_{t-\ell}, \\
\Xi_{U,it}(h) &= \sum_{\ell=1}^{h} \mathbb{1}\{t-\ell \geq 1\} \hat{s}_i \delta_{i,h-\ell} X_{t-\ell}.
\end{aligned}$$

Under Assumption 2, it can be readily verified that $\{\chi_{T,t}(h, \kappa_T)\}_{t=1}^{T}$ is a martingale difference array adapted to the natural filtration $\{\mathcal{F}_{T,t}\}_{t=1}^{T}$,

$$\mathcal{F}_{T,t} = \sigma\left(\{X_\tau, Z_\tau, \{u_{i\tau}\}_{i=1}^{N}\}_{\tau \leq t}, \{\theta_i, s_i\}_{i=1}^{N} \right),$$

that is, $\chi_{T,t}(h, \kappa_T)$ is $\mathcal{F}_{T,t}$-measurable and $E_{\kappa_T}\left[\chi_{T,t}(h, \kappa_T) \middle| \mathcal{F}_{T,t-1}\right] = 0$.

By construction, $\sum_{t=1}^{T} E_{\kappa_T}\left[\chi_{T,t}(h, \kappa_T)^2\right] = 1$ and we can show (Lemmas 4 and 5)

$$\sum_{t=1}^{T} \chi_{T,t}(h, \kappa_T)^2 \xrightarrow[P_{\kappa_T}]{\text{p}} 1 \text{ and } \lim_{T\to\infty} \sum_{t=1}^{T} E_{\kappa_T}\left[\chi_{T,t}(h, \kappa_T)^4\right] = 0.$$

By Davidson (1994, Theorems 23.11, 23.16 and 24.3), the Lemma follows. □

Lemma 2 (Consistency of the standard error).

$$\frac{\hat{V}(h)}{V(h, \kappa_T)} \xrightarrow[P_{\kappa_T}]{\text{p}} 1.$$

Proof. Since $V(h, \kappa_T) > 0$ holds P_{κ_T}-a.s., it suffices to show that

$$\frac{\hat{V}(h) - V(h, \kappa_T)}{g(\kappa_T^2/N)} \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Write

$$\frac{\hat{V}(h) - V(h, \kappa_T)}{g(\kappa_T^2/N)} = D_{T,1}(h, \kappa_T) + D_{T,2}(h, \kappa_T),$$

where we have defined

$$D_{T,1}(h, \kappa_T) = \sum_{t=1}^{T-h} \frac{\left(X_t^2 \xi_t(h, \kappa_T)^2 - E_{\kappa_T}\left[X_t^2 \xi_t(h, \kappa_T)^2 \middle| \{\theta_i, s_i\}_{i=1}^{N}\right]\right)}{(T-h) g(\kappa_T^2/N)},$$

$$D_{T,2}(h, \kappa_T) = \sum_{t=1}^{T-h} \left[\frac{\left(N^{-1} \sum_{i=1}^{N} \hat{x}_{it}(h) \hat{\xi}_{it}(h)\right)^2 - X_t^2 \xi_t(h, \kappa_T)^2}{(T-h) g(\kappa_T^2/N)}\right].$$

Next, using $(x^2 - y^2) = (x - y)(x + y)$ and the Cauchy-Schwarz inequality,

$$\left|D_{T,2}(h, \kappa_T)\right| \leq \sqrt{D_{T,2}^{-}(h, \kappa_T)} \sqrt{D_{T,2}^{+}(h, \kappa_T)},$$

with

$$D_{T,2}^{-}(h, \kappa_T) = \sum_{t=1}^{T-h} \frac{\left[\left(N^{-1} \sum_{i=1}^{N} \hat{x}_{it}(h) \hat{\xi}_{it}(h)\right) - X_t \xi_t(h, \kappa_T)\right]^2}{(T-h) g(\kappa_T^2/N)},$$

$$D^+_{T,2}(h,\kappa_T) = \sum_{t=1}^{T-h} \frac{\left[\left(N^{-1}\sum_{i=1}^N \hat{x}_{it}(h)\hat{\xi}_{it}(h)\right) + X_t\xi_t(h,\kappa_T)\right]^2}{(T-h)g(\kappa_T^2/N)}.$$

Adding and subtracting $X_t\xi_t(h,\kappa_T)$ within the squares in $D^+_{T,2}(h,\kappa_T)$ and applying Loève's inequality,

$$D^+_{T,2}(h,\kappa_T) \leq 2D^-_{T,2}(h,\kappa_T) + 8|D_{T,1}(h,\kappa_T)| + \frac{8V(h,\kappa_T)}{g(\kappa_T^2/N)}.$$

We can show (Lemmas 6 and 7) that $D_{T,1}(h,\kappa_T) = o_{P_{\kappa_T}}(1)$ and $D^-_{T,2}(h,\kappa_T) = o_{P_{\kappa_T}}(1)$. Given that $V(h,\kappa_T)/g(\kappa_T^2/N)$ is bounded P_{κ_T}-a.s., $D^+_{T,2}(h,\kappa_T) = O_{P_{\kappa_T}}(1)$ which implies $D_{T,2}(h,\kappa_T) = o_{P_{\kappa_T}}(1)$ and the Lemma follows. □

Lemma 3 (Negligibility of the reminder).

$$R_T(h,\kappa_T) \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Proof. Let $\bar{x}_t(h) = (X_{t-1} - \bar{X}_1(h), \ldots, X_{t-p} - \bar{X}_p(h))'$ where $\bar{X}_\ell(h) = (T-h)^{-1}\sum_{t=1}^{T-h} X_{t-\ell}$. Since either $\hat{s}_i$ was demeaned or time effects were not included as controls,

$$\hat{\pi}(h)'W_{it} = \hat{\pi}_{0,i}(h) + \sum_{\ell=1}^{p} \hat{\pi}_{X,\ell}(h)\hat{s}_i X_{t-\ell} = \hat{s}_i\left(\bar{X}_0(h) + \hat{\pi}_X(h)'\bar{x}_t(h)\right),$$

where $\{\hat{\pi}_{0,i}(h)\}$, $\pi_X(h) = (\hat{\pi}_{X,1}(h), \ldots, \hat{\pi}_{X,p}(h))'$ are the coefficients from the regression of $s_i X_t$ on unit fixed effects and p lags of $\hat{s}_i X_t$. Furthermore, it is readily seen that $\hat{\pi}_X(h)$ are also the coefficients in a regression of X_t on $\bar{x}_t(h)$,

$$\hat{\pi}_X(h) = \left[\sum_{t=1}^{T-h} \bar{x}_t(h)\bar{x}_t(h)'\right]^{-1} \sum_{t=1}^{T-h} \bar{x}_t(h)X_t.$$

Note that $E\left[X_{t-\ell}\right] = E\left[X_{t-\ell}X_t\right] = 0$ and that $\text{Var}\left(\sum_{t=1}^{T-h} X_{t-\ell}\right), \text{Var}\left(\sum_{t=1}^{T-h} X_{t-\ell}X_t\right)$ are bounded by a constant (M_2 and M_4, respectively) times $(T-h)$ under Assumptions 1, 2 and 3. Also note that $(T-h)^{-1}\sum_{t=1}^{T-h} \bar{x}_t(h)\bar{x}_t(h)' = E\left[X_t^2\right] \times I_p + o_{P_{\kappa_T}}(1)$. All of this is independent of κ_T. It follows that

$$\bar{X}_0(h) = O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right), \quad \hat{\pi}_X(h) = O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right).$$

Write

$$R_T(h,\kappa_T) = -\frac{\bar{X}_0(h)\sum_{t=1}^{T-h}\xi_t(h,\kappa_T)}{\sqrt{(T-h)V(h,\kappa_T)}} - \frac{\hat{\pi}_X(h)'\sum_{t=1}^{T-h}\bar{x}_t(h)\xi_t(h,\kappa_T)}{\sqrt{(T-h)V(h,\kappa_T)}}.$$

To obtain $R_T(h,\kappa_T) = o_{P_{\kappa_T}}(1)$, we show $\{(T-h)V(h,\kappa_T)\}^{-1/2}\sum_{t=1}^{T}\xi_t(h,\kappa_T) = O_{P_{\kappa_T}}(1)$ and $\{(T-h)V(h,\kappa_T)\}^{-1/2}\sum_{t=1}^{T}\bar{x}_t(h)\xi_t(h,\kappa_T) = O_{P_{\kappa_T}}(1)$. We do so by direct calculation.

First,

$$\begin{aligned}
E_{N,\kappa_T}\left[\left(\sum_{t=1}^{T-h}\xi_t(h,\kappa_T)\right)^2\right] &= E_N\left[\left(\sum_{t=1}^{T-h}\sum_{\ell=0}^{\infty}\iota_\ell(h)\bar{\beta}_\ell X_{t+h-\ell}\right)^2\right] + E_N\left[\left(\sum_{t=1}^{T-h}\sum_{\ell=0}^{\infty}\bar{\gamma}_\ell Z_{t+h-\ell}\right)^2\right] \\
&\quad + \frac{\kappa_T^2}{N^2}E_N\left[\left(\sum_{t=1}^{T-h}\sum_{i=1}^{N}\sum_{\ell=0}^{\infty}\hat{s}_i\delta_{i\ell}u_{i,t+h-\ell}\right)^2\right] \\
&\le 2(T-h)\Bigg[\left(\sum_{\ell=0}^{\infty}\iota_\ell(h)|\bar{\beta}_\ell|\right)^2 E_N\left[X_t^2\right] + \left(\sum_{\ell=0}^{\infty}|\bar{\gamma}_\ell|\right)^2 E_N\left[Z_t^2\right] \\
&\quad + \frac{\kappa_T^2}{N^2}\sum_{i=1}^{N}\left(\sum_{\ell=0}^{\infty}|\hat{s}_i\delta_{i\ell}|\right)^2 E_N\left[u_{it}^2\right]\Bigg] \\
&\le (T-h)\times 2(2+\kappa_T^2/N)C^4M_2,
\end{aligned}$$

where the last line uses Assumption 3(i)–(iv).[1] By iterated expectations and Chebyshev's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}
P_{\kappa_T}\left(\left|\frac{\sum_{t=1}^{T}\xi_t(h,\kappa_T)}{\sqrt{(T-h)V(h,\kappa_T)}}\right| > \varepsilon\right) &= E_{\kappa_T}\left[P_{N,\kappa_T}\left(\left|\frac{\sum_{t=1}^{T}\xi_t(h,\kappa_T)}{\sqrt{(T-h)V(h,\kappa_T)}}\right| > \varepsilon\right)\right] \\
&\le \frac{1}{\varepsilon^2}E_{\kappa_T}\left[\frac{2(2+\kappa_T^2/N)C^4M_2}{V(h,\kappa_T)}\right] \le \frac{1}{\varepsilon^2}\frac{6C^4M_2}{\underline{CM}^2} < \infty,
\end{aligned}$$

[1] We also used the fact that for any linear process $\omega_t = \sum_{\ell=0}^{\infty}\varphi_\ell\varepsilon_{t-\ell}$ where $\{\varphi_\ell\}$ are absolutely summable and $\{\varepsilon_t\}$ is white noise with $E[\varepsilon_t] = 0$ and $E\left[\varepsilon_t^2\right] = 1$,

$$E\left[\left(\sum_{t=1}^{T}\omega_t\right)^2\right] = \sum_{m=-(T-1)}^{T-1}(T-|m|)\sum_{\ell=0}^{\infty}\varphi_\ell\varphi_{\ell+|m|} \le T\sum_{\ell=0}^{\infty}|\varphi_\ell|\sum_{m=-\infty}^{\infty}|\varphi_{\ell+|m|}| \le 2T\left(\sum_{\ell=0}^{\infty}|\varphi_\ell|\right)^2.$$

where the bound on $(2+\kappa_T^2/N)/V(h,\kappa_T) = ((2+\kappa_T^2/N)/g(\kappa_T^2/N)) \times (g(\kappa_T^2/N)/V(h,\kappa_T))$ uses $(2+\kappa)/g(\kappa) \leq 3$ and $V(h,\kappa_T)/g(\kappa_T^2/N) \geq \underline{\mathrm{C}}\mathrm{M}^2$.

Similarly for any $k = 1, \ldots, p$,

$$\begin{aligned} E_{N,\kappa_T}\left[\left(\sum_{t=1}^{T-h} X_{t-k}\xi_t(h,\kappa_T)\right)^2\right] &\leq (T-h)\left[\sum_{\ell=0}^{\infty} \iota_\ell(h)\bar{\beta}_\ell^2 E_N\left[X_{t-k}^2 X_{t+h-\ell}^2\right] + \sum_{\ell=0}^{\infty} \bar{\gamma}_\ell^2 E_N\left[X_{t-k}^2 Z_{t+h-\ell}^2\right]\right. \\ &\quad \left. + \frac{\kappa_T^2}{N^2}\sum_{i=1}^{N}\sum_{\ell=0}^{\infty} \hat{s}_i^2\delta_{i\ell}^2 E_N\left[X_{t-k}^2 u_{i,t+h-\ell}^2\right] + 2\sum_{\ell=1}^{h+k} \iota_{h+k-\ell}(h)\iota_{h+k+\ell}(h)|\bar{\beta}_{h+k-\ell}\bar{\beta}_{h+k+\ell}| E_N\left[X_{t-k}^2 X_{t-k-\ell}^2\right]\right] \\ &\leq (T-h) \times (4+\kappa_T^2/N)C^4 M_4, \end{aligned}$$

where we used the autocovariances of $X_{t-k}\xi_t(h,\kappa_T)$ and Assumption 3(i)–(iv) again. By iterated expectations and Chebyshev, for any $\varepsilon > 0$,

$$P_{\kappa_T}\left(\left|\frac{\sum_{t=1}^T X_{t-r}\xi_t(h,\kappa_T)}{\sqrt{(T-h)V(h,\kappa_T)}}\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} E_{\kappa_T}\left[\frac{(4+\kappa_T^2/N)C^4 M_4}{V(h,\kappa_T)}\right] \leq \frac{1}{\varepsilon^2}\frac{5C^4 M_4}{\underline{\mathrm{C}}\mathrm{M}^2} < \infty.$$

Thus, $R_T(h,\kappa_T) = o_{P_{\kappa_T}}(1)$ and the Lemma follows. ∎

Lemma 4. *Under the conditions of Lemma 1,*

$$\sum_{t=1}^{T} \chi_{T,t}(h,\kappa_T)^2 \xrightarrow[P_{\kappa_T}]{\mathrm{p}} 1.$$

Proof. We show $\mathrm{Var}_{N,\kappa_T}\left(\sum_{t=1}^T \chi_{T,t}(h,\kappa_T)^2\right) \leq \bar{V}/(T-h)$ for a constant $\bar{V}$ independent of κ_T. Since $E_{N,\kappa_T}\left[\sum_{t=1}^T \chi_{T,t}(h,\kappa_T)^2\right] = 1$, by iterated expectations and Chebyshev's inequality, for any $\varepsilon > 0$,

$$\begin{aligned} P_{\kappa_T}\left(\left|\sum_{t=1}^T \chi_{T,t}(h,\kappa_T)^2 - 1\right| > \varepsilon\right) &= E_{\kappa_T}\left[P_{N,\kappa_T}\left(\left|\sum_{t=1}^T \chi_{T,t}(h,\kappa_T)^2 - 1\right| > \varepsilon\right)\right] \\ &\leq \frac{\bar{V}}{\varepsilon^2(T-h)} \to 0. \end{aligned}$$

As argued at the beginning of the section, $V(h,\kappa)/g(\kappa^2/N)$ is bounded away from zero and infinity uniformly over κ. Thus, it suffices to show

$$\mathrm{Var}_{N,\kappa_T}\left(\sum_{t=1}^T \frac{V(h,\kappa_T)\chi_{T,t}(h,\kappa_T)^2}{g(\kappa_T^2/N)}\right) \leq \frac{\bar{V}}{T-h},$$

P_{κ_T}-a.s., for some constant $\bar{V}$ independent of κ_T. We do this by a direct calculation. Define $\bar{\chi}_{T,t}(h,\kappa_T) = \chi_{T,t}(h,\kappa_T)\left\{(T-h)V(h,\kappa_T)/g(\kappa_T^2/N)\right\}^{1/2}$ so that

$$
\begin{aligned}
g\left(\frac{\kappa_T}{\sqrt{N}}\right)\bar{\chi}_{T,t}(h,\kappa_T) &= X_t\Xi_{X,t}(h,\kappa) + Z_t\Xi_{Z,t}(h) + \frac{\kappa_T}{N}\sum_{i=1}^{N} u_{it}\Xi_{U,it}(h) \\
&= \underbrace{\sum_{\ell=1}^{\infty} b_{t,\ell}X_tX_{t-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{1,t}} + \underbrace{\sum_{\ell=0}^{\infty} c_{t,\ell}X_tZ_{t-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{2,t}} + \underbrace{\sum_{\ell=1}^{h} \tilde{c}_{t,\ell}Z_tX_{t-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{3,t}} \\
&\quad + \underbrace{\frac{\kappa_T}{N}\sum_{i=1}^{N}\sum_{\ell=0}^{\infty} d_{it,\ell}X_tu_{i,t-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{4,t}} + \underbrace{\frac{\kappa_T}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{h} \tilde{d}_{it,\ell}u_{it}X_{t-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{5,t}} \qquad \text{(B.1)}
\end{aligned}
$$

for some $\{b_{t,\ell}, c_{t,\ell}, \tilde{c}_{t,\ell}, \{d_{it,\ell}, \tilde{d}_{it,\ell}\}_{i=1}^N\}$ that depend on $\{\theta_i, s_i\}_{i=1}^N$ (and h). Note that the coefficients depend on t only via the indicator functions $\mathbb{1}\{t-\ell \le 1\}$ and $\mathbb{1}\{t \le T-h\}$. It will be convenient to define $\{b_\ell, c_\ell, \tilde{c}_\ell, \{d_{i,\ell}, \tilde{d}_{i,\ell}\}_{i=1}^N\}$ as the coefficients we would get by setting the indicators to one. This implies $|b_{t,\ell}| \le |b_\ell|$, $|c_{t,\ell}| \le |c_\ell|$, and so on. By Assumption 3(iv), $|b_\ell|, |c_\ell|, |\tilde{c}_\ell|, |d_{i\ell}|, |\tilde{d}_{i\ell}| \le \bar{C}_\ell$ almost surely for finite constants $\bar{C}_\ell$ such that $\bar{C} = \sum_{\ell=1}^\infty \bar{C}_\ell < \infty$ (in fact, we can take $\bar{C} \le C^2$ independent of h).

Consider the variance

$$
\mathrm{Var}_{N,\kappa_T}\left(\sum_{t=1}^{T}\frac{V(h,\kappa_T)\chi_{T,t}(h,\kappa_T)^2}{g(\kappa_T^2/N)}\right) = \frac{\sum_{t=1}^T\sum_{\tau=1}^T \Gamma_T(t,\tau)}{(T-h)^2}
$$

where (omitting the dependence on h, κ_T and $\{\theta_i, s_i\}_{i=1}^N$)

$$
\Gamma_T(t,\tau) = \mathrm{Cov}_{N,\kappa_T}\left(\tilde{\chi}_{T,t}(h,\kappa_T)^2, \tilde{\chi}_{T,\tau}(h,\kappa_T)^2\right).
$$

Expanding the square of $\tilde{\chi}_{T,t}(h,\kappa_T)$ and using the linearity of the covariance we can express $\Gamma_T(t,\tau)$ as the sum of covariances $\Gamma_{T,k_1k_2k_3k_4}(t,\tau) = \mathrm{Cov}_{N,\kappa_T}\left(\zeta_{k_1,t}\zeta_{k_2,t}, \zeta_{k_3,\tau}\zeta_{k_4,\tau}\right)$ where k_1, k_2, k_3, k_4 range over the five terms in (B.1). Moreover, if $k_1 = k_2$, $\Gamma_{T,k_1k_2k_3k_4}(t,\tau)$ can only be non-zero if $k_3 = k_4$, while if $k_1 \ne k_2$, only if either $k_1 = k_3$

and $k_2 = k_4$ or $k_1 = k_4$ and $k_2 = k_3$. Then, by the triangle inequality,

$$\begin{aligned}
\left|\Gamma_T(t,\tau)\right| &= \left|\sum_{k_1=1}^{5}\sum_{k_2=1}^{5}\sum_{k_3=1}^{5}\sum_{k_4=1}^{5}\Gamma_{T,k_1k_2k_3k_4}(t,\tau)\right| \\
&\leq \sum_{k_1=1}^{5}\sum_{k_3=1}^{5}\left|\Gamma_{T,k_1k_1k_3k_3}(t,\tau)\right| + 2\sum_{k_1=1}^{5}\sum_{k_2=1}^{5}\left|\Gamma_{T,k_1k_2k_1k_2}(t,\tau)\right|. \qquad \text{(B.2)}
\end{aligned}$$

We begin with $\sum_{t=1}^{T}\sum_{\tau=1}^{T}\Gamma_{T,k_1k_1k_3k_3}(t,\tau)$. Consider $k_1 = k_3 = 1$:

$$\begin{aligned}
g\left(\frac{\kappa_T^4}{N^2}\right)\left|\Gamma_{T,1111}(t,\tau)\right| &= \left|\mathrm{Cov}_N\left(\left(\sum_{\ell=1}^{\infty} b_{t,\ell}X_tX_{t-\ell}\right)^2, \left(\sum_{\ell=1}^{\infty} b_{\tau,\ell}X_\tau X_{\tau-\ell}\right)^2\right)\right| \\
&= \left|\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=1}^{\infty}\sum_{\ell_4=1}^{\infty} b_{t,\ell_1}b_{t,\ell_2}b_{\tau,\ell_3}b_{\tau,\ell_4}\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}X_{t-\ell_2}, X_\tau^2X_{\tau-\ell_3}X_{\tau-\ell_4}\right)\right| \\
&\leq \sum_{\ell_1=1}^{\infty}\sum_{\ell_3=1}^{\infty} b_{\ell_1}^2 b_{\ell_3}^2\left|\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}^2, X_\tau^2X_{\tau-\ell_3}^2\right)\right| \\
&\quad + 2\sum_{\ell_1=1}^{\infty}\sum_{\ell_2\neq\ell_1}|b_{\ell_1}b_{\ell_2}b_{\ell_1+\tau-t}b_{\ell_2+\tau-t}|\left|\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}X_{t-\ell_2}, X_\tau^2X_{t-\ell_1}X_{t-\ell_2}\right)\right|.
\end{aligned}$$

The inequality uses the fact that by Assumption 2, $\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}X_{t-\ell_2}, X_\tau^2X_{\tau-\ell_3}X_{\tau-\ell_4}\right)$ can only be non-zero if $\ell_1 = \ell_2$ and $\ell_3 = \ell_4$ or, with $\ell_1 \neq \ell_2$, if either $\ell_3 = \ell_1 + \tau - t$ and $\ell_4 = \ell_2 + \tau - t$ or $\ell_3 = \ell_2 + \tau - t$ and $\ell_4 = \ell_1 + \tau - t$.[2] We also use $|b_{t,\ell}| \leq |b_\ell|$.

For the first double sum, now summing over t and τ,

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{\ell_1=1}^{\infty}\sum_{\ell_3=1}^{\infty} b_{\ell_1}^2 b_{\ell_3}^2\left|\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}^2, X_\tau^2X_{\tau-\ell_3}^2\right)\right| \\
&\qquad \leq 2T\sum_{m=0}^{T-1}\sum_{\ell_1=1}^{\infty}\sum_{\ell_3=1}^{\infty}\bar{C}_{\ell_1}^2\bar{C}^2\left|\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}^2, X_{t-m}^2X_{t-m-\ell_3}^2\right)\right| \\
&\qquad \leq 2T\bar{C}^2\sum_{\ell_1=1}^{\infty}\bar{C}_{\ell_1}^2\left(\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\left|\mathrm{Cov}_N\left(X_t^2X_{t-\ell_1}^2, X_{t-m}^2X_{t-m-\ell_3}^2\right)\right|\right)
\end{aligned}$$

[2]This is similar to the proof of Montiel Olea and Plagborg-Møller (2021, Lemma A.6)

$$\leq 2T\bar{C}^2\bar{K}\sum_{\ell_1=1}^{\infty}\bar{C}_{\ell_1}^2 \leq 2T\bar{C}^4\bar{K}$$

for some constant $\bar{K}$ that can be shown to exist as by Assumption 3(iii) the fourth-order cumulants of X_t^2 conditional on $\{\theta_i, s_i\}_{i=1}^N$ are absolutely summable.

Turning to the second double sum, by Assumption 2, since $\ell_1 \neq \ell_2$,

$$\left|\mathrm{Cov}_N\!\left(X_t^2 X_{t-\ell_1}X_{t-\ell_2}, X_\tau^2 X_{\tau-\ell_1}X_{\tau-\ell_2}\right)\right| = \left|E_N\left[X_t^2X_\tau^2X_{t-\ell_1}^2X_{t-\ell_2}^2\right]\right| \leq E_N\left[X_t^8\right] \leq M_8,$$

where M_8 is the moment bound from Assumption 3(i). Then,

$$\begin{aligned}
2\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2\neq\ell_1} &|b_{\ell_1}b_{\ell_2}b_{\ell_1+\tau-t}b_{\ell_2+\tau-t}|\left|\mathrm{Cov}_N\!\left(X_t^2X_{t-\ell_1}X_{t-\ell_2}, X_\tau^2X_{t-\ell_1}X_{t-\ell_2}\right)\right| \\
&\leq 4TM_8\sum_{m=0}^{T-1}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2\neq\ell_1}|b_{\ell_1}b_{\ell_2}b_{\ell_1+m}b_{\ell_2+m}| \\
&\leq 4TM_8\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}|b_{\ell_1}||b_{\ell_2}|\left(\sum_{m=0}^{\infty}|b_{\ell_1+m}||b_{\ell_2+m}|\right) \\
&\leq 4TM_8\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}|b_{\ell_1}||b_{\ell_2}|\left(\sum_{m_1=1}^{\infty}|b_{m_1}|^2\sum_{m_2=1}^{\infty}|b_{m_2}|^2\right)^{1/2} \\
&\leq 4T\bar{C}^4M_8,
\end{aligned}$$

where the second inequality increases the range of summation over ℓ_2 and m, the third uses Cauchy-Schwarz and the fourth follows from Assumption 3(iv).

Putting these calculations together and using $g(\kappa) \geq 1$,

$$\frac{\sum_{t=1}^T\sum_{\tau=1}^T\left|\Gamma_{T,1111}(t,\tau)\right|}{(T-h)^2} \leq \frac{T\times 2\bar{C}^4(\bar{K}+2M_8)}{g(\kappa_T^4/N^2)(T-h)^2} \leq \frac{2\bar{C}^4(\bar{K}+2M_8)}{(1-h/T)(T-h)}.$$

In fact, the same bound works for $\sum_{t=1}^T\sum_{\tau=1}^T\left|\Gamma_{T,k_1k_1k_3k_3}(t,\tau)\right|$ for any $k_1, k_3 \in \{1,2,3\}$.

Next consider $k_1 = k_3 = 4$:

$$g\left(\frac{\kappa_T^4}{N^2}\right)\frac{\left|\Gamma_{T,4444}(t,\tau)\right|}{(\kappa_T^4/N^4)} = \left|\mathrm{Cov}_N\!\left(\left(\sum_{i=1}^N\sum_{\ell=1}^{\infty}d_{it,\ell}X_tu_{i,t-\ell}\right)^2, \left(\sum_{i=1}^N\sum_{\ell=1}^{\infty}d_{i\tau,\ell}X_\tau u_{i,\tau-\ell}\right)^2\right)\right|$$

$$= \left| \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \sum_{i_3=1}^{N} \sum_{i_4=1}^{N} \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sum_{\ell_3=1}^{\infty} \sum_{\ell_4=1}^{\infty} d_{i_1 t,\ell_1} d_{i_2 t,\ell_2} d_{i_3 \tau,\ell_3} d_{i_4 \tau,\ell_4} \times \mathrm{Cov}_N\Big(X_t^2 u_{i_1,t-\ell_1} u_{i_2,t-\ell_2}, X_\tau^2 u_{i_3,\tau-\ell_3} u_{i_4,\tau-\ell_4} \Big) \right|$$

$$\leq \sum_{i_1=1}^{N} \sum_{i_3=1}^{N} \left| \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sum_{\ell_3=1}^{\infty} \sum_{\ell_4=1}^{\infty} d_{i_1 t,\ell_1} d_{i_1 t,\ell_2} d_{i_3 \tau,\ell_3} d_{i_3 \tau,\ell_4} \times \mathrm{Cov}_N\Big(X_t^2 u_{i_1,t-\ell_1} u_{i_1,t-\ell_2}, X_\tau^2 u_{i_3,\tau-\ell_3} u_{i_3,\tau-\ell_4} \Big) \right|$$

$$+ \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \left| \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sum_{\ell_3=1}^{\infty} \sum_{\ell_4=1}^{\infty} d_{i_1 t,\ell_1} d_{i_2 t,\ell_2} d_{i_1 \tau,\ell_3} d_{i_2 \tau,\ell_4} \times \mathrm{Cov}_N\Big(X_t^2 u_{i_1,t-\ell_1} u_{i_2,t-\ell_2}, X_\tau^2 u_{i_1,\tau-\ell_3} u_{i_2,\tau-\ell_4} \Big) \right|$$

$$+ \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \left| \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=1}^{\infty} \sum_{\ell_3=1}^{\infty} \sum_{\ell_4=1}^{\infty} d_{i_1 t,\ell_1} d_{i_2 t,\ell_2} d_{i_2 \tau,\ell_3} d_{i_1 \tau,\ell_4} \times \mathrm{Cov}_N\Big(X_t^2 u_{i_1,t-\ell_1} u_{i_2,t-\ell_2}, X_\tau^2 u_{i_2,\tau-\ell_3} u_{i_1,\tau-\ell_4} \Big) \right|.$$

The inequality uses the fact that $\mathrm{Cov}_N\Big(X_t^2 u_{i_1,t-\ell_1} u_{i_1,t-\ell_2}, X_\tau^2 u_{i_3,\tau-\ell_3} u_{i_3,\tau-\ell_4} \Big)$ can only be non-zero if $i_1 = i_2$ and $i_3 = i_4$, or $i_1 = i_3$ and $i_2 = i_4$, or $i_1 = i_4$ and $i_2 = i_3$.

Summing over t and τ and applying to each of the three summands on the right hand side the same steps as the case $k_1 = k_3 = 1$,

$$\frac{\sum_{t=1}^{T} \sum_{\tau=1}^{T} \left| \Gamma_{T,4444}(t,\tau) \right|}{(T-h)^2} \leq \frac{3N^2 \times \kappa_T^4/N^4 \times 2\bar{C}^4(\bar{K}+2M_8)}{g(\kappa_T^4/N^2)(1-h/T)(T-h)} \leq \frac{6\bar{C}^4(\bar{K}+2M_8)}{(1-h/T)(T-h)}.$$

Repeating the calculation for the remaining cases (and noting that this bound is three times larger than the one we computed for $k_1 = k_3 = 1$) we conclude that $6\bar{C}^4(\bar{K}+2M_8)/(1-h/T)(T-h)$ works for any $k_1, k_3 \in \{1,2,3,4,5\}$. By similar reasoning, the bound also works for $\sum_{t=1}^{T} \sum_{\tau=1}^{T} \Gamma_{T,k_1 k_2 k_1 k_2}(t,\tau)$ whenever $k_1 \neq k_2$. We then get

$$\frac{\sum_{t=1}^{T} \sum_{\tau=1}^{T} \Gamma_T(t,\tau)}{(T-h)^2} \leq \frac{\bar{V}}{(T-h)},$$

where $\bar{V} = 75 \times 6\bar{C}^4(\bar{K} + 2M_8)/(1 - h/T)$ does not depend on κ_T (75 is the number of covariances in (B.2)). This establishes $\sum_{t=1}^{T} \chi_{T,t}(h, \kappa_T)^2 = 1 + o_{P_{\kappa_T}}(1)$. □

Lemma 5. *Under the conditions of Lemma 1,*

$$\lim_{T\to\infty} \sum_{t=1}^{T} E_{\kappa_T}\left[\chi_{T,t}^4\right] = 0.$$

Proof. Using the notation of Lemma 4 and Loève's inequality,

$$E_N\left[\bar{\chi}_{T,t}(h, \kappa_T)^4\right] \leq 5^3 \sum_{k=1}^{5} E_N\left[\zeta_{k,t}^4\right]. \tag{B.3}$$

Each of the five terms in (B.3) is under Assumption 3(i)–(iv) bounded by a constant that does not depend on κ_T. For $k = 1$,

$$\begin{aligned}
g\left(\frac{\kappa_T^4}{N^2}\right) E_N\left[\zeta_{1,t}^4\right] &= E_N\left[\left(\sum_{\ell=1}^{\infty} b_{t,\ell} X_t X_{t-\ell}\right)^4\right] \\
&\leq \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=1}^{\infty}\sum_{\ell_4=1}^{\infty} |b_{t,\ell_1} b_{t,\ell_2} b_{t,\ell_3} b_{t,\ell_4}| \left|E_N\left[X_t^4 X_{t-\ell_1} X_{t-\ell_2} X_{t-\ell_3} X_{t-\ell_4}\right]\right| \\
&\leq M_8 \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=1}^{\infty}\sum_{\ell_4=1}^{\infty} |b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4}| \leq M_8 \left(\sum_{\ell=1}^{\infty} |b_\ell|\right)^4 \leq M_8 \bar{C}^4,
\end{aligned}$$

where $\bar{C}$ is the constant we defined in the first part. The same bound works for $k = 2$ and $k = 3$ in (B.3). For $k = 4$,

$$\begin{aligned}
g\left(\frac{\kappa_T^4}{N^2}\right) \frac{E_N\left[\zeta_{4,t}^4\right]}{(\kappa_T^4/N^4)} &= E_N\left[\left(\sum_{i=1}^{N}\sum_{\ell=1}^{\infty} d_{it,\ell} X_t u_{i,t-\ell}\right)^4\right] \\
&\leq \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N}\sum_{i_4=1}^{N}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=1}^{\infty}\sum_{\ell_4=1}^{\infty} |d_{i_1t,\ell_1} d_{i_2t,\ell_2} d_{i_3t,\ell_3} d_{i_4t,\ell_4}| \\
&\quad \times \left|E_N\left[X_t^4 u_{i_1,t-\ell_1} u_{i_2,t-\ell_2} u_{i_3,t-\ell_3} u_{i_4,t-\ell_4}\right]\right| \\
&\leq 3 \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} |d_{i_1t,\ell_1}^2 d_{i_2t,\ell_2}^2| \left|E_N\left[X_t^4 u_{i_1,t-\ell_1}^2 u_{i_2,t-\ell_2}^2\right]\right|
\end{aligned}$$

$$\leq 3N^2 M_8 \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} d_{\ell_1}^2 d_{\ell_2}^2 \leq 3N^2 M_8 \bar{C}^4,$$

where the second inequality uses that for $E_N\left[X_t^4 u_{i_1,t-\ell_1} u_{i_2,t-\ell_2} u_{i_3,t-\ell_3} u_{i_4,t-\ell_4}\right]$ to be non-zero we need $i_1 = i_2$ and $i_3 = i_4$, or $i_1 = i_3$ and $i_2 = i_4$, or $i_1 = i_4$ and $i_2 = i_3$ because of Assumptions 1(ii) and 2. The same bound applies to $k = 5$ in (B.3).

Putting these bounds together,

$$\sum_{t=1}^{T} E_N\left[\chi_{T,t}(h,\kappa_T)^4\right] = \frac{\sum_{t=1}^{T} E_N\left[\bar{\chi}_{T,t}(h,\kappa_T)^4\right] g(\kappa_T^4/N^2)}{(T-h)^2 V(h,\kappa_T)^2} \leq \frac{9M_8\bar{C}^4 g(\kappa_T^4/N^2)}{(1-h/T)(T-h)V(h,\kappa_T^2)^2}.$$

Since $V(h,\kappa_T)^2/g(\kappa_T^4/N^2) \geq \underline{\mathrm{CM}}^2 > 0$, using iterated expectations we conclude that $\sum_{t=1}^{T} E_{\kappa_T}\left[\chi_{T,t}(h,\kappa_T)^4\right] = o(1)$ where the convergence is uniform over κ_T. ◻

Lemma 6. *Under the conditions of Lemma 2,*

$$\sum_{t=1}^{T-h} \frac{X_t^2\xi_t(h,\kappa_T)^2 - E_{\kappa_T}\left[X_t^2\xi_t(h,\kappa_T)^2 \middle| \{\theta_i, s_i\}_{i=1}^N\right]}{(T-h)g(\kappa_T^2/N)} \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Proof. The proof is analogous to that of Lemma 4. We will show that for a constant $\bar{V}$ independent of κ_T, $\mathrm{Var}_{N,\kappa_T}\left(\sum_{t=1}^{T} X_t^2\xi_t(h,\kappa_T)^2/g(\kappa_T^2/N)\right) \leq \bar{V}(T-h)$. By iterated expectations and Chebyshev's inequality it will follow that, for any $\varepsilon > 0$,

$$P_{\kappa_T}\left(\left|\sum_{t=1}^{T} \frac{X_t^2\xi_t(h,\kappa_T)^2 - E_{N,\kappa_T}\left[X_t^2\xi_t(h,\kappa_T)^2\right]}{(T-h)g(\kappa_T^2/N)}\right| > \varepsilon\right) \leq \frac{\bar{V}}{\varepsilon^2(T-h)} \to 0.$$

We can write

$$\begin{aligned} X_t\xi_t(h,\kappa_T) &= \sum_{\ell=0}^{\infty} \iota_\ell(h)\bar{\beta}_\ell X_t X_{t+h-\ell} + \sum_{\ell=0}^{\infty} \bar{\gamma}_\ell X_t Z_{t+h-\ell} + \frac{\kappa_T}{N}\sum_{i=1}^{N}\sum_{\ell=0}^{\infty} \hat{s}_i \delta_{i\ell} X_t u_{i,t-\ell} \\ &= \underbrace{\sum_{\ell=0}^{\infty} b_\ell X_t X_{t+h-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{1,t}} + \underbrace{\sum_{\ell=0}^{\infty} c_\ell X_t Z_{t+h-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{2,t}} + \underbrace{\frac{\kappa_T}{N}\sum_{i=1}^{N}\sum_{\ell=0}^{\infty} d_{i\ell} X_t u_{i,t+h-\ell}}_{\equiv g(\kappa_T/\sqrt{N})\zeta_{3,t}}. \end{aligned} \quad \text{(B.4)}$$

for some coefficients $\{b_\ell, c_\ell, \{d_{i\ell}\}_{i=1}^N\}$ that depend on $\{\theta_i, s_i\}_{i=1}^N$ (and h). By Assumption 3(iv), we have $|b_\ell|, |c_\ell|, |d_{i\ell}| \leq C_\ell$ almost surely for some positive finite constants C_ℓ

such that $C = \sum_{\ell=1}^{\infty} C_\ell < \infty$. Note that the coefficients, constants and variables $\zeta_{1,t}$, $\zeta_{2,t}$, $\zeta_{3,t}$ are different from the ones in the proof of Lemma 4.

Consider the variance

$$\mathrm{Var}_{N,\kappa_T}\left(\sum_{t=1}^{T} \frac{X_t^2 \xi_t(h,\kappa_T)^2}{g(\kappa_T^2/N)}\right) = \sum_{t=1}^{T-h}\sum_{\tau=1}^{T-h} \Gamma_T(t,\tau)$$

where (omitting the dependence on h, κ_T and $\{\theta_i, s_i\}_{i=1}^N$)

$$\Gamma_T(t,\tau) = \mathrm{Cov}_{N,\kappa_T}\left(\frac{X_t^2 \xi_t(h,\kappa_T)^2}{g(\kappa_T/\sqrt{N})}, \frac{X_\tau^2 \xi_\tau(h,\kappa_T)^2}{g(\kappa_T/\sqrt{N})}\right).$$

As in the proof of Lemma 4, we expand the square of $X_t^2 \xi_t(h,\kappa_T)^2$ to express $\Gamma_T(t,\tau)$ as the sum of covariances $\Gamma_{T,k_1k_2k_3k_4}(t,\tau) = \mathrm{Cov}_{N,\kappa_T}\left(\zeta_{k_1,t}\zeta_{k_2,t}, \zeta_{k_3,\tau}\zeta_{k_4,\tau}\right)$ where k_1, k_2, k_3, k_4 range over the three terms in (B.4). If $k_1 = k_2$, $\Gamma_{T,k_1k_2k_3k_4}(t,\tau)$ can only be non-zero if $k_3 = k_4$, while if $k_1 \neq k_2$, only if either $k_1 = k_3$ and $k_2 = k_4$ or $k_1 = k_4$ and $k_2 = k_3$. Then,

$$\begin{aligned}\left|\Gamma_T(t,\tau)\right| &= \sum_{k_1=1}^{3}\sum_{k_2=1}^{3}\sum_{k_3=1}^{3}\sum_{k_4=1}^{3} \left|\Gamma_{T,k_1k_2k_3k_4}(t,\tau)\right| \\ &= \sum_{k_1=1}^{3}\sum_{k_3=1}^{3} \left|\Gamma_{T,k_1k_1k_3k_3}(t,\tau)\right| + 2\sum_{k_1=1}^{3}\sum_{k_2=1}^{3} \left|\Gamma_{T,k_1k_2k_1k_2}(t,\tau)\right|. \qquad \text{(B.5)}\end{aligned}$$

By calculations similar to that of Lemma 4, for any $k_1, k_2, k_3 \in \{1,2,3\}$,

$$\begin{aligned}\sum_{t=1}^{T-h}\sum_{\tau=1}^{T-h} \left|\Gamma_{T,k_1k_1k_3k_3}(t,\tau)\right| &\le 6C^4(\bar{K} + 2M_8) \times (T-h), \\ \sum_{t=1}^{T-h}\sum_{\tau=1}^{T-h} \left|\Gamma_{T,k_1k_2k_1k_2}(t,\tau)\right| &\le 6C^4(\bar{K} + 2M_8) \times (T-h).\end{aligned}$$

We therefore arrive at

$$\sum_{t=1}^{T}\sum_{\tau=1}^{T} \Gamma_T(t,\tau) \le \bar{V}(T-h),$$

with $\bar{V} = 27 \times 6C^4(\bar{K} + 2M_8)$ independent of κ_T (27 is the number of terms in (B.5)). Hence, $\{(T-h)g(\kappa_T^2/N)\}^{-1} \sum_{t=1}^{T} \left(X_t^2 \xi_t(h,\kappa_T)^2 - E_{N,\kappa_T}\left[X_t^2 \xi_t(h,\kappa_T)^2\right]\right) = o_{P_{\kappa_T}}(1)$. □

Lemma 7. *Under the conditions of Lemma 2,*

$$\sum_{t=1}^{T-h} \frac{\left[\left(N^{-1}\sum_{i=1}^{N}\hat{x}_{it}(h)\hat{\xi}_{it}(h)\right) - X_t\xi_t(h,\kappa_T)\right]^2}{(T-h)g(\kappa_T^2/N)} \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Proof. We begin by writing

$$\hat{x}_{it}(h) = \hat{s}_i(X_t - \hat{X}_t(h)), \quad \hat{X}_t(h) = \bar{X}_0(h) + \hat{\pi}_X(h)'\bar{x}_t(h), \tag{B.6}$$

with $\bar{X}_0(h)$, $\hat{\pi}_X(h)$ and $\bar{x}_t(h) = (X_{t-1} - \bar{X}_1(h), \ldots, X_{t-p} - \bar{X}_p(h))'$ as in the proof of Lemma 3. As argued, $\bar{X}_0(h) = O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right)$ and $\hat{\pi}_X(h) = O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right)$.

Next, we write $\hat{\eta}(h)'W_{it} = \hat{\eta}_{0,i}(h) + \hat{\eta}_X(h)'\bar{x}_t(h)\hat{s}_i$ and $\eta_{X,ih} = (\beta_{i,h+1}, \ldots, \beta_{i,h+p})'$ so that

$$\hat{\xi}_{it}(h) - \xi_{it}(h,\kappa_T) = \left(\mu_i - \hat{\eta}_{0,i}(h) + \sum_{\ell=1}^{p}\beta_{i,h+\ell}\bar{X}_\ell(h)\right) + (\beta_{ih} - \hat{\beta}(h)\hat{s}_i)X_t + (\eta_{X,ih} - \hat{\eta}_X(h)\hat{s}_i)'\bar{x}_t(h)$$

and we note

$$\begin{aligned}\begin{pmatrix}\hat{\beta}(h)\\ \hat{\eta}_X(h)\end{pmatrix} &= \left[\sum_{t=1}^{T-h}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}'\right]^{-1}\sum_{t=1}^{T-h}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}\hat{Y}_{t+h}\\ &= \begin{pmatrix}\tilde{\beta}(h)\\ \tilde{\eta}_X(h)\end{pmatrix} + \left[\sum_{t=1}^{T-h}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}'\right]^{-1}\sum_{t=1}^{T-h}\begin{pmatrix}X_t - \bar{X}_0(h)\\ \bar{x}_t(h)\end{pmatrix}\frac{\xi_t(h,\kappa_T)}{(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2)}\end{aligned}$$

where $\hat{Y}_{t+h} = (\sum_{i=1}^{N}\hat{s}_i^2)^{-1}\sum_{i=1}^{N}\hat{s}_iY_{i,t+h}$ and $\tilde{\eta}_X(h) = (\sum_{i=1}^{N}\hat{s}_i^2)^{-1}\sum_{i=1}^{N}\hat{s}_i\eta_{X,ih}$. Since the least squares denominator matrix when scaled by $(T-h)^{-1}$ converges to $E\left[X_t^2\right] \times I_{p+1}$ in probability uniformly over κ_T, the calculations in Lemma 3 imply that

$$\begin{aligned}\frac{(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2)(\hat{\beta}(h) - \tilde{\beta}(h))}{g(\kappa_T/\sqrt{N})} &= O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right),\\ \frac{(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2)(\hat{\eta}_X(h) - \tilde{\eta}_X(h))}{g(\kappa_T/\sqrt{N})} &= O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right).\end{aligned}$$

Because W_{it} includes unit effects, $\sum_{i=1}^{N}\hat{x}_{it}(h)(\hat{\eta}_{0,i}(h) - \mu_i + \sum_{\ell=1}^{p}\beta_{i,h+\ell}\bar{X}_\ell(h)) = 0$ and,

$$N^{-1}\sum_{i=1}^{N}\hat{x}_{it}(h)(\hat{\xi}_{it}(h) - \xi_{it}(h,\kappa_T)) = \left(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2\right)(\tilde{\beta}(h) - \hat{\beta}(h))X_t(X_t - \hat{X}_t(h))$$

$$+\left(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2\right)(\tilde{\eta}_X(h)-\hat{\eta}_X(h))'\bar{x}_t(h)(X_t-\hat{X}_t(h)). \tag{B.7}$$

To prove the Lemma, add and subtract $N^{-1}\sum_{i=1}^{N}\hat{x}_{it}(h)\xi_{it}(h,\kappa_T)$ within the squares and use Loève's inequality to obtain

$$\sum_{t=1}^{T-h}\frac{\left[\left(N^{-1}\sum_{i=1}^{N}\hat{x}_{it}(h)\hat{\xi}_{it}(h)\right)-X_t\xi_t(h,\kappa_T)\right]^2}{(T-h)g(\kappa_T^2/N)}\leq 2D_{T,2}^{\pi}(h,\kappa_T)+2D_{T,2}^{\eta}(h,\kappa_T),$$

where

$$D_{T,2}^{\pi}(h,\kappa_T)=\sum_{t=1}^{T-h}\frac{\left[N^{-1}\sum_{i=1}^{N}(\hat{s}_iX_t-\hat{x}_{it}(h))\xi_{it}(h,\kappa_T)\right]^2}{(T-h)g(\kappa_T^2/N)},$$

$$D_{T,2}^{\eta}(h,\kappa_T)=\sum_{t=1}^{T-h}\frac{\left[N^{-1}\sum_{i=1}^{N}\hat{x}_{it}(h)(\hat{\xi}_{it}(h)-\xi_{it}(h,\kappa_T))\right]^2}{(T-h)g(\kappa_T^2/N)}.$$

Inserting (B.6) into the first term and using Loève's inequality,

$$D_{T,2}^{\pi}(h,\kappa_T)\leq 2\left[\bar{X}_0(h)^2\frac{\sum_{t=1}^{T-h}\xi_t(h,\kappa_T)^2}{(T-h)g(\kappa_T^2/N)}+\|\hat{\pi}_X(h)\|^2\frac{\sum_{t=1}^{T-h}\|\bar{x}_t(h)\xi_t(h,\kappa_T)\|^2}{(T-h)g(\kappa_T^2/N)}\right],$$

where $\|\cdot\|$ is Euclidean norm. From calculations similar to those in Lemma 3,

$$\frac{\sum_{t=1}^{T-h}\xi_t(h,\kappa_T)^2}{(T-h)g(\kappa_T^2/N)}=O_{P_{\kappa_T}}(1)\ \text{and}\ \frac{\sum_{t=1}^{T-h}\|\bar{x}_t(h)\xi_t(h,\kappa_T)\|^2}{(T-h)g(\kappa_T^2/N)}=O_{P_{\kappa_T}}(1),$$

which allows us to conclude that $D_{T,2}^{\pi}(h,\kappa_T)=o_{P_{\kappa_T}}(1)$.

Inserting (B.7) into the second term and using Loève's inequality,

$$D_{T,2}^{\eta}(h,\kappa_T)\leq 2\left[\left(\frac{(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2)(\tilde{\beta}(h)-\hat{\beta}(h))}{g(\kappa_T/\sqrt{N})}\right)^2\frac{\sum_{t=1}^{T-h}X_t^2(X_t-\hat{X}_t(h))^2}{T-h}\right.$$
$$\left.+\left\|\left(\frac{(N^{-1}\sum_{i=1}^{N}\hat{s}_i^2)(\tilde{\eta}_X(h)-\hat{\eta}_X(h))}{g(\kappa_T/\sqrt{N})}\right)\right\|^2\frac{\sum_{t=1}^{T-h}\|\bar{x}_t(h)(X_t-\hat{X}_t(h))\|^2}{T-h}\right].$$

Under Assumption 3(i), we can show that $(T-h)^{-1}\sum_{t=1}^{T-h}X_t^2(X_t-\hat{X}_t(h))^2=O_{P_{\kappa_T}}(1)$ and $(T-h)^{-1}\sum_{t=1}^{T-h}\|x_t(h)(X_t-\hat{X}_t(h))\|^2=O_{P_{\kappa_T}}(1)$. Thus, $D_{T,2}^{\eta}(h,\kappa_T)=o_{P_{\kappa_T}}(1)$. □

Proposition 2

Parts (A), (B) and (C) of the proof of Proposition 2 in Appendix A are established in Lemmas 8, 9 and 10 below. The argument closely resembles the proof of Proposition 1 and, therefore, in order to conserve space we only sketch the steps. Again, we adopt Assumptions 1, 2 and 3, we fix p and assume $h_T/T \leq \phi < 1$ as $T, N \to \infty$.

Lemma 8 (Asymptotic normality of the score).

$$\frac{\sum_{t=1}^{T-h_T} X_t \xi_t(h_T, \kappa_T)}{\sqrt{(T-h_T)V(h_T, \kappa_T)}} \xrightarrow[P_{\kappa_T}]{\mathrm{d}} N(0,1).$$

Proof. The proof given for Lemma 1 goes through with the following adjustment: we can remove the terms $\bar{\beta}_\ell, \bar{\gamma}_\ell, \delta_{i\ell}$ from $\Xi_{X,t}(h,\kappa)$ whenever $\ell > h$. That is, we set

$$\Xi_{X,t}(h,\kappa) = \sum_{\ell=1}^{h} \mathbb{1}\{t-\ell \geq 1\}\bar{\beta}_{h-\ell} X_{t-\ell} + \mathbb{1}\{t \leq T-h\}\left[\bar{\gamma}_h Z_t + \frac{\kappa}{N}\sum_{i=1}^{N} \hat{s}_i \delta_{ih} u_{it}\right].$$

The calculations in Lemmas 4 and 5 apply with the same adjustment. In Lemma 4, $\bar{V} \leq 75 \times 6C^8(\bar{K} + 2M_8)/(1-\phi)$, which does not depend on κ_T or h_T. Similarly, in Lemma 5, $\sum_{t=1}^{T} E_N\left[\chi_{T,t}(h_T,\kappa_T)^4\right] \leq 9M_8C^8/(1-\phi)^2\underline{\mathrm{CM}}^2 T$, which tends to zero as $T \to \infty$ uniformly over κ_T and h_T. □

Lemma 9 (Consistency of the standard error).

$$\frac{\hat{V}(h_T)}{V(h_T,\kappa_T)} \xrightarrow[P_{\kappa_T}]{\mathrm{p}} 1.$$

Proof. The proofs of Lemma 2 and auxiliary Lemma 6 go through without change. To establish the equivalent to Lemma 7 in this context, define $\bar{x}_t(h_T)$ as in its proof and let $\bar{y}_{it}(h_T) = (\hat{Y}_{i,t-1}(h_T), \ldots, \hat{Y}_{i,t-p}(h_T))$ with $\hat{Y}_{i,t-\ell}(h_T)$ the residual from regressing $g(\kappa_T)^{-1} Y_{i,t-\ell}$ on unit and time effects. We can write

$$\hat{\pi}(h_T)' W_{it} = \hat{s}_i \bar{X}_0(h_T) + \hat{s}_i \hat{\pi}_X(h_T)' \bar{x}_t(h_T) + \hat{\pi}_Y(h_T)' \bar{y}_{it}(h_T),$$
$$\hat{\eta}(h_T)' W_{it} = \hat{\eta}_{0,i}(h_T) + \hat{s}_i \hat{\eta}_X(h_T)' \bar{x}_t(h_T) + \hat{\eta}_Y(h_T)' \bar{y}_{it}(h_T).$$

Scaling $Y_{i,t-\ell}$ by $g(\kappa_T)^{-1}$ leaves the least square predictions $\hat{\pi}(h_T)' W_{it}$ and $\hat{\eta}(h_T)' W_{it}$ unchanged, but it helps bound them in probability uniformly over κ_T.

Calculations similar to those in Lemma 3 deliver

$$\begin{pmatrix} \bar{X}_0(h_T) \\ \hat{\pi}_X(h_T) \\ \hat{\pi}_Y(h_T) \end{pmatrix} = O_{P_{\kappa_T}}\left((T-h_T)^{-1/2}\right),$$

$$g\left(\frac{\kappa_T}{\sqrt{N}}\right)^{-1} \begin{pmatrix} (\hat{\beta}(h_T) - \tilde{\beta}(h_T)) \\ (\hat{\eta}_X(h_T) - \tilde{\eta}_X(h_T)) \\ (\hat{\eta}_Y(h_T) - \tilde{\eta}_Y(h_T)) \end{pmatrix} = O_{P_{\kappa_T}}\left((T-h_T)^{-1/2}\right),$$

where $\tilde{\eta}_X(h_T) = (\tilde{B}_1(h_T), \ldots, \tilde{B}_p(h_T))'$ and $\tilde{\eta}_Y(h_T) = g(\kappa_T)(A_1(h_T), \ldots, A_p(h_T))'$ with $A_\ell(h)$ and $\tilde{B}_\ell(h)$ as defined in the proof of Proposition 2 in Appendix A.

The rest of the proof follows the steps of Lemma 7. The convergence is uniform in both κ_T and h_T because $T - h_T \leq (1-\phi)T$ with $\phi < 1$. □

Lemma 10 (Negligibility of the remainder).

$$R_T(h_T, \kappa_T) \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Proof. We begin by defining $\bar{x}_t(h_T)$ and $\bar{y}_{it}(h)$ as in Lemma 9, by writing

$$\hat{\pi}(h_T)' W_{it} = \hat{s}_i \bar{X}_0(h_T) + \hat{s}_i \hat{\pi}_X(h_T)' \bar{x}_t(h_T) + \hat{\pi}_Y(h_T)' \bar{y}_{it}(h_T),$$

and by noting again that

$$\begin{pmatrix} \bar{X}_0(h_T) \\ \hat{\pi}_X(h_T) \\ \hat{\pi}_Y(h_T) \end{pmatrix} = O_{P_{\kappa_T}}\left((T-h_T)^{-1/2}\right).$$

Next, we write $r_{it}(h_T) = (\beta_{ih} - \tilde{\beta}(h)\hat{s}_i)X_t + \sum_{\ell=1}^{p}(B_{i\ell}(h) - \tilde{B}_\ell(h)\hat{s}_i)X_{t-\ell}$ and

$$R_T(h_T, \kappa_T) = -\frac{\bar{X}_0(h_T)\sum_{t=1}^{T-h_T} \xi_t(h_T, \kappa_T)}{\sqrt{(T-h_T)V(h_T, \kappa_T)}} - \frac{\hat{\pi}_X(h_T)' \sum_{t=1}^{T-h_T} \bar{x}_t(h_T)\xi_t(h_T, \kappa_T)}{\sqrt{(T-h_T)V(h_T, \kappa_T)}} - \frac{\hat{\pi}_Y(h_T)' \sum_{i=1}^{N}\sum_{t=1}^{T-h_T} \bar{y}_{it}(h_T)(r_{it}(h_T) + \xi_{it}(h_T, \kappa_T))}{N\sqrt{(T-h_T)V(h_T, \kappa_T)}}$$

The rest of the argument mimics the proof of Lemma 3. □

Proposition 3

Parts (A), (B) and (C) of the proof of Proposition 3 in Appendix A are stated in Lemmas 11, 12 and 13 below. The proofs are virtually identical to their counterparts in Proposition 1 with some minor differences. Here we make Assumptions 4 and we hold h and $p \geq h$ fixed as $T, N \to \infty$.

Lemma 11 (Asymptotic normality of the score).

$$\frac{\sum_{t=1}^{T-h} \boldsymbol{\lambda}' \boldsymbol{X}_t^* \xi_t(h, \kappa_T)}{\sqrt{(T-h)\boldsymbol{\lambda}' \boldsymbol{V}(h, \kappa_T) \boldsymbol{\lambda}}} \xrightarrow[P_{\kappa_T}]{\text{d}} N(0,1).$$

Proof. The arguments given for Lemma 1 and auxiliary Lemmas 4 and 5 apply with the obvious change in notation. □

Lemma 12 (Consistency of the standard error and OLS denominator).

$$\frac{\boldsymbol{\lambda}' \hat{\boldsymbol{V}}^{\text{IV}}(h) \boldsymbol{\lambda}}{\boldsymbol{\lambda}' \boldsymbol{V}(h, \kappa_T) \boldsymbol{\lambda}} \xrightarrow[P_{\kappa_T}]{\text{p}} 1 \text{ and } \hat{\boldsymbol{J}}^{\text{IV}}(h) \xrightarrow[P_{\kappa_T}]{\text{p}} \boldsymbol{J}.$$

Proof. The first part follows from arguments analogous to those given for Lemma 2 and auxiliary Lemmas 6 and 7 (with obvious notational changes). For the second part, note $\text{Var}_{N,\kappa_T}\left(X_t^* \tilde{X}_t\right) \leq \bar{V}/(T-h)$ for some constant $\bar{V}$ independent of κ_T under Assumption 4(ii), so that $\left\| \hat{\boldsymbol{J}}^{\text{IV}}(h) - \boldsymbol{J} \right\| = o_{P_{\kappa_T}}(1)$ follows from iterated expectations and Chebyshev's inequality. □

Lemma 13 (Negligibility of the remainder).

$$R_T(h, \kappa_T) \xrightarrow[P_{\kappa_T}]{\text{p}} 0.$$

Proof. For any $\boldsymbol{\lambda} \neq 0_{(p+1)\times 1}$, by the same calculations as in Lemma 3,

$$\frac{\sum_{t=1}^{T-h} \boldsymbol{\lambda}' \boldsymbol{X}_t^*}{(T-h)} = O_{P_{\kappa_T}}\left((T-h)^{-1/2}\right) \text{ and } \frac{\sum_{t=1}^{T-h} \xi_t(h, \kappa_T)}{\sqrt{(T-h)\boldsymbol{\lambda}' \boldsymbol{V}(h, \kappa_T) \boldsymbol{\lambda}}} = O_{P_{\kappa_T}}(1)\,.$$

Since $\hat{\boldsymbol{J}}^{\text{IV}}(h) = \boldsymbol{J} + o_{P_{\kappa_T}}(1)$ by the second part of Lemma 12, the result follows. □

C Details of simulation study

Here we complement Section 4 with additional details. First, we describe how we simulate the heterogeneity. Second, we specify the calibration of our DGPs. Third and last, we present further simulation results.

Simulation of observable and unobservable heterogeneity. A primary feature is the correlation between s_i and $\{\beta_i, \gamma_i.\delta_i\}$.[3] We begin by drawing the vector

$$(s_i, s_{\gamma,i}, s_{\delta,i})' \sim N\left(1_{3\times1}, (1-\rho)I_3 + \rho 1_{3\times3}\right)$$

for some $\rho \neq 0$. Next, we set a very large $\bar{L}$ and compute

$$\beta_{i\ell} = s_i\breve{\beta}_{i\ell}, \quad \gamma_{i\ell} = s_{\gamma,i}\breve{\gamma}_{i\ell}, \quad \delta_{i\ell} = s_{\delta,i}\breve{\delta}_{i\ell},$$

where $\{\breve{\beta}_{i\ell}, \breve{\gamma}_{i\ell}, \breve{\delta}_{i\ell}\}_{\ell=0}^{\bar{L}}$ are obtained by (a) drawing the roots of ARMA polynomials from Beta distributions, (b) computing their MA(∞) representations, (c) truncating them at $\bar{L}$, and (d) normalizing them so that $\sum_{\ell=0}^{\bar{L}} \breve{\beta}_{i\ell}^2 = \sum_{\ell=0}^{\bar{L}} \breve{\gamma}_{i\ell}^2 = \sum_{\ell=0}^{\bar{L}} \breve{\delta}_{i\ell}^2 = 1$.[4]

To generate time-varying heterogeneity we set $s_{it} = s_i + \zeta_{it}$ with $\zeta_{it} \sim N(0,1)$, i.i.d. over units and time, and independent of s_i and everything else. This ensures s_{it} remains exogenous with respect to aggregate and idiosyncratic shocks.

Finally, in the VAR DGP, we set

$$B_{i\ell} = s_i\breve{B}_{i\ell}, \quad C_{i0} = s_{\gamma,i}, \quad D_{i0} = s_{\delta,i}.$$

where $\{\breve{B}_{i\ell}\}_{\ell=0}^{\bar{L}}$ are obtained in the same way as $\{\breve{\beta}_{i\ell}\}_{\ell=0}^{\bar{L}}$ above.

Our method does not satisfy Assumption 3(iv), although responses are bounded with sufficiently high probability that it does not seem to make a difference.

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[3]Instead, μ_i (and m_i in the VAR setup) does not play a big role and we simply draw it as $N(0,1)$.

[4]The advantage of this representation is that it separates the scale and persistence. For example, if X_t is white noise with unit variance conditional on $\{\beta_{i\ell}\}_{\ell=0}^{\bar{L}}$, the variance of $\sum_{\ell=0}^{\bar{L}} \beta_{i\ell}X_{t-\ell}$ is $\sum_{\ell=0}^{\bar{L}} \beta_{i\ell}^2 = s_i^2$ while the ratio of long-run variance to variance of $\sum_{\ell=0}^{\bar{L}} \beta_{i\ell}X_{t-\ell}$ (a measure of persistence) is

$$\frac{\left(\sum_{\ell=0}^{\bar{L}} \beta_{i\ell}\right)^2}{\sum_{\ell=0}^{\bar{L}} \beta_{i\ell}^2} = \frac{\left(\sum_{\ell=0}^{\bar{L}} \breve{\beta}_{i\ell}\right)^2}{\sum_{\ell=0}^{\bar{L}} \breve{\beta}_{i\ell}^2},$$

which does not depend on s_i.

DGP calibration. In the general DGP, we set $\rho = 0.5$, and generate $\{\breve{\beta}_{i\ell}, \breve{\gamma}_{i\ell}, \breve{\delta}_{i\ell}\}_{\ell=0}^{\bar{L}}$ from ARMA(4, 2) processes with expected roots $(0.7, 0.3, 0.2, 0.1)$ and $(0, 0)$ for $\breve{\beta}_{i\ell}$, $(0.7, 0.2, 0.1, -0.2)$ and $(0.2, -0.2)$ for $\breve{\gamma}_{i\ell}$, and $(0.9, 0.3, 0.1, 0.1)$ and $(0.5, 0.2)$ for $\breve{\delta}_{i\ell}$. We draw each root as $\text{Beta}(\bar{\lambda}\nu, (1-\bar{\lambda})\nu)$ where $\bar{\lambda}$ is the mean listed above and $\nu = 10$, and we truncate polynomials at $\bar{L} = 2T$ lags.

In the LP-IV case, we use a similar method for $\{b_\ell, c_\ell\}_{\ell=0}^{\bar{L}}$. We obtain b_ℓ from an ARMA(1, 1) with roots 0.3 and −0.2, and c_ℓ from an ARMA(2, 2) with roots $(0.4, 0.2)$ and $(0.1, -0.1)$. We also set $a_0 = 10$ to be safely above standard weak IV thresholds.

Finally, for the VAR DGP, we draw $\{\breve{B}_{i\ell}\}_{\ell=0}^{p}$ from an MA(2) with roots $(0.8, -0.5)$ and $\nu = 10$, and we set $\{A_\ell\}_{\ell=1}^{p}$ to an AR(2) with roots $(1 - 5/T, 0.5)$.

The mean and quantiles of responses for each horizon can be seen in Figure C.1.

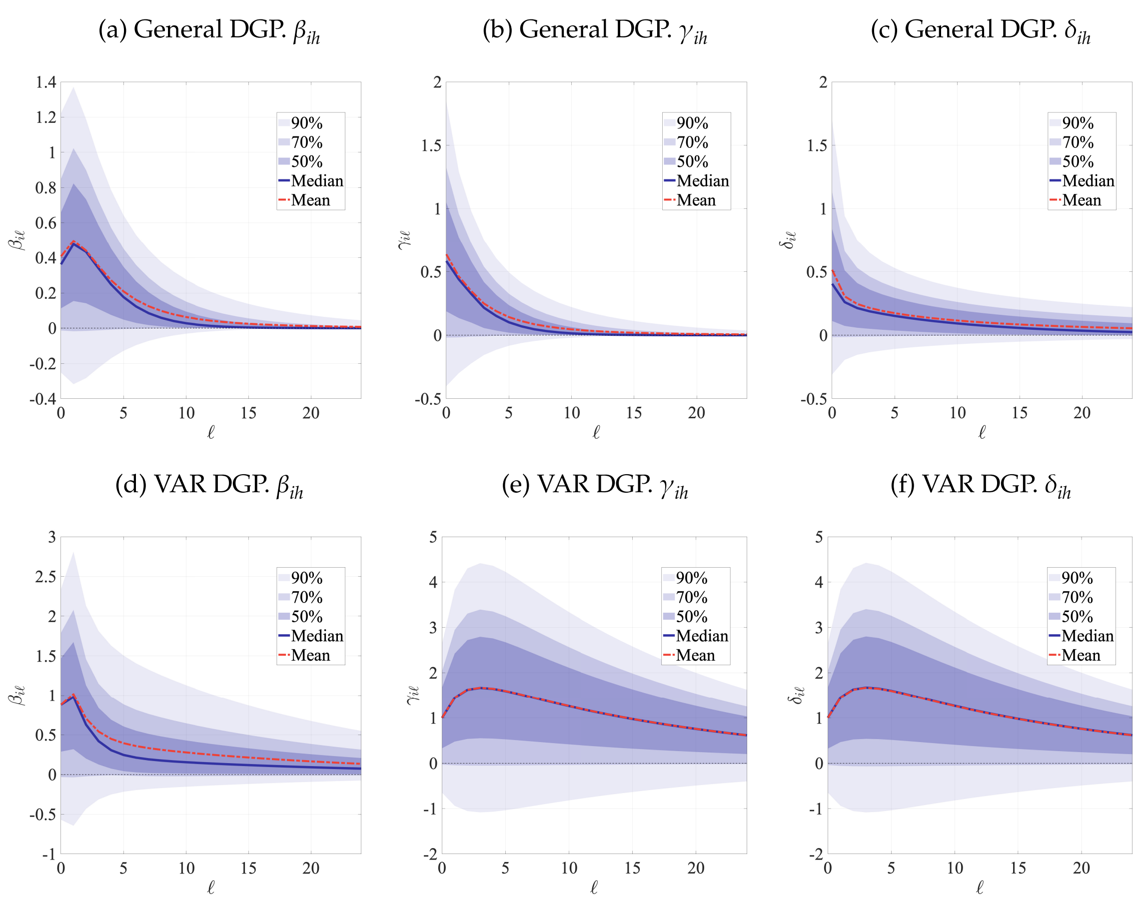

FIGURE C.1. Distributions of impulse responses across ℓ for general and VAR DGPs.

Additional results. Figure C.2 presents coverage rates of 90% confidence intervals in the general DGP with $T = 100$ for panel LPs on X_t (panels (a)-to-(c)) and on $s_{it}X_t$ (panels (d)-to-(f)).[5] As mentioned in the paper, the estimands are different: LPs on X_t recover the mean impulse response while LPs on $s_{it}X_t$ recover their projection on s_{it}. Yet, the observations we made about inference from Section 4 are unchanged. In particular, t-LAHR inference dominates all the alternatives in delivering correct coverage for the nonparametric panel local projection estimand.

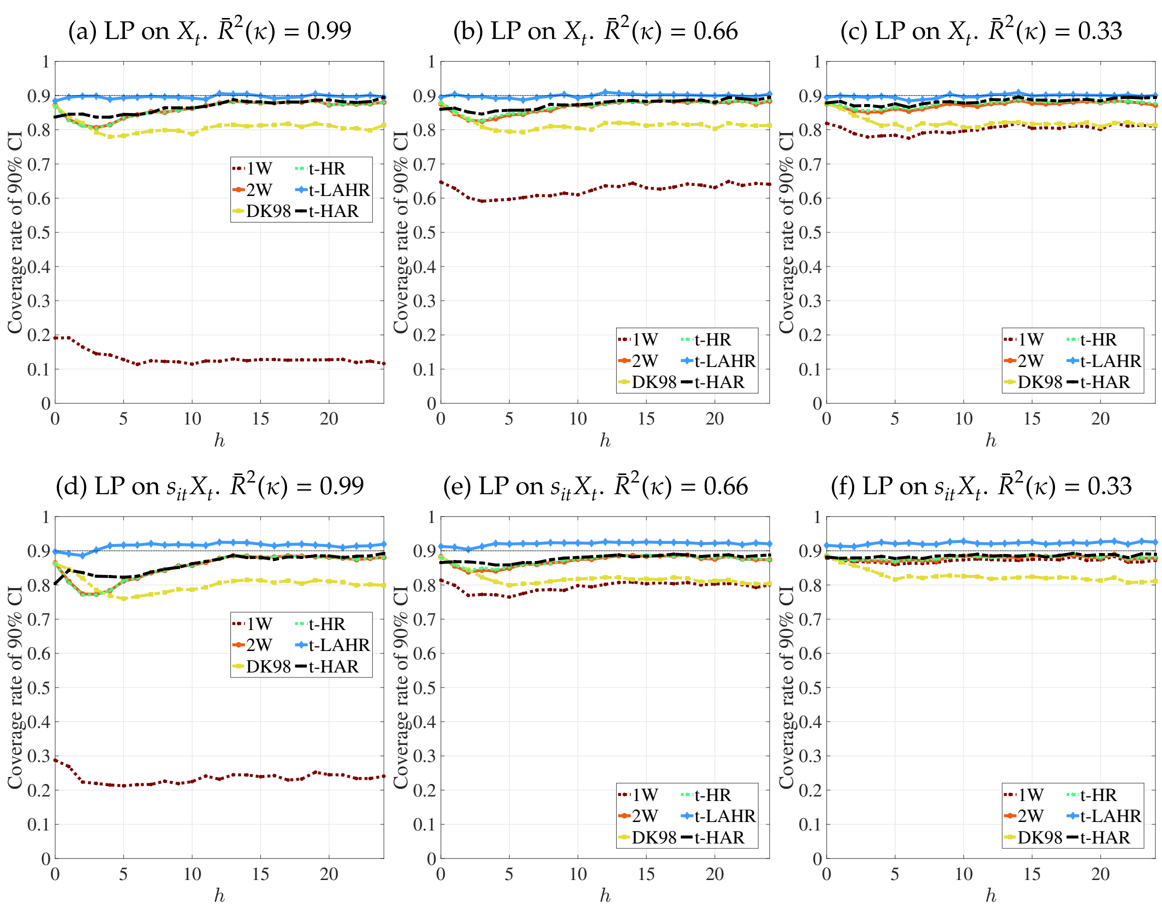

FIGURE C.2. Coverage rates of 90% confidence intervals for $T = 100$.

Note: 1W refers to one-way (unit-level) clustering, 2W to two-way clustering, DK98 to Driscoll–Kraay, and t-HR/t-LAHR/t-HAR to the time-level clustering approaches discussed in the text.

[5] For panel LPs on X_t time effects are excluded from the vector of controls. Otherwise, the estimation and inference procedures are the same as in Figure 1 in the paper.

D A survey of empirical applications

Below, we survey relevant empirical applications by the method used to calculate standard errors. The list reflects the recent surge in applications (with the oldest paper dated 2018) and includes both published work and working papers. We have aimed to make the list comprehensive, but it is possible that some might have been inadvertently omitted. When different methods were used, we favored the one used in the main specification and the one used in estimation of dynamic effects (non-zero horizons). We classified as one-way clustering (within units) applications that cluster at a higher level of aggregation than primary units; say, at the industry (or industry-time) level when units are firms. While allowing for sector-level shocks, these still rule out economy-wide spatial dependence. See the Introduction for additional details.

By method	
Two-way clustering (within units and time)	Ippolito, Ozdagli, and Perez-Orive (2018), Jeenas (2019), Ottonello and Winberry (2020), Amberg, Jansson, Klein, and Rogantini Picco (2022), Palazzo and Yamarthy (2022), Paz (2022), Bellifemine, Couturier, and Jamilov (2023), Cascaldi-Garcia, Vukotić, and Zubairy (2023), Drechsel (2023), Durante, Ferrando, and Vermeulen (2022), Duval, Furceri, Lee, and Tavares (2023), Ferreira, Ostry, and Rogers (2023), González, Nuño, Thaler, and Albrizio (2023), Lakdawala and Moreland (2023), Singh, Suda, and Zervou (2023), Thürwächter (2023), Zhou (2023), Anderson and Cesa-Bianchi (2024), Berthold, Cesa-Bianchi, Di Pace, and Haberis (2024), Caglio, Darst, and Kalemli-Özcan (2024), Camêlo (2024), Gulyas, Meier, and Ryzhenkov (2024), Paranhos (2024), Lakdawala and Moreland (forthcoming)
Clustering within units	Wu (2018), Ozdagli (2018), Crouzet and Mehrotra (2020), Singh, Suda, and Zervou (2022), Albrizio, González, and Khametshin (2023), Andersen, Johannesen, Jørgensen, and Peydró (2023), Camara and Ramirez Venegas (2023), Ghomi (2023), Indarte (2023), Bardóczy, Bornstein, Maggi, and Salgado (2024), Jeenas (2024), Jeenas and Lagos (2024), Lo Duca, Moccero, and Parlapiano (2024), Paranhos (2024), Ruzzier (2024)
Driscoll and Kraay (1998) standard errors	Holm, Paul, and Tischbirek (2021), Bahaj, Foulis, Pinter, and Surico (2022), Cloyne, Ferreira, Froemel, and Surico (2023), Fagereng, Gulbrandsen, Holm, and Natvik (2023), Gorea, Kryvtsov, and Kudlyak (2023), Bilal and Känzig (2024), Cao, Hegna, Holm, Juelsrud, König, and Riiser (2024)
Clustering within time	Gürkaynak, Karasoy-Can, and Lee (2022)

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